Linear Algebra, Winter 2022

List 7

More complex numbers, intro to polynomials

164. Re-write $(r\,e^{i\theta})^3$ in the form $\_\_\,e^{\_\_ i}$.

$$e^{\theta i} = \cos(\theta) + i\sin(\theta)$$

165. Re-write $10e^{(\pi/4)i}$ in the form $\_\_ + \_\_ i$.

166. Re-write $(2e^{7i})^{10}$ in the form $\_\_ + \_\_ i$.

167. Re-write $-\sqrt{5} + \sqrt{15}\,i$ in the form $\_\_ e^{\_\_ i}$.

168. If z is a complex number with $|z| = 4$, what is $|z^2|$?

169. If z is a complex number with $\arg(z) = 5\pi/6$, what is $\arg(z^2)$?

170. If w is a complex number with $\arg(w) = \pi/10$, what is $\arg(w^{446})$?

171. Write $\left(\dfrac{\sqrt{3}-i}{1+i}\right)^6$ in the form $a + bi$.

(Hint: $\sqrt{3} - i = 2e^{(-\pi/6)i}$ and $1 + i = \sqrt{2}e^{(\pi/4)i}$.)

Rectangular form: $a + b\,i$, or $a + i\,b$, or $b\,i + a$, or similar.
Polar form: $r\big(\cos\theta + i\sin\theta\big)$, or $r\cos(\theta) + r\sin(\theta)\,i$, or similar. Requires $r \geq 0$.
Exponential form: $r\,e^{\theta\,i}$, or $r\,e^{i\,\theta}$.

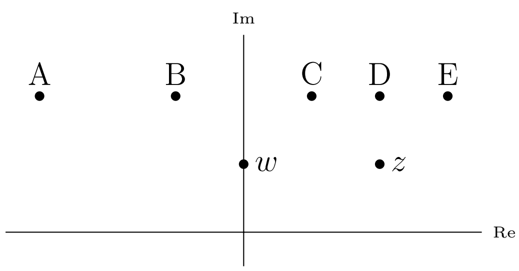
172. Write the following in rectangular form.

(a) $e^{\frac{\pi}{4}i}$ (b) $2e^{i\pi/6}$ (c) $5e^{-i\pi/3}$ (d) $-8e^{\pi i}$ (e) $\sqrt{9}+\sqrt{-9}$

173. For $z = 1 + i$ and $w = 3e^{(\pi/4)i}$, calculate the following. For complex values, you may give the answer in rectangular or polar or exponential form (your choice).

(a) $|w|$
(b) $|zw|$
(c) $|z/w|$
(d) $-w$
(e) $(\bar{w})^2$
(f) zw
(g) $z + w$
(h) $z + z^2$
(i) z/w
(j) w/z

174. Which of the points A-E below could be $z + w$?



175. Which of the points A-E above could be zw?

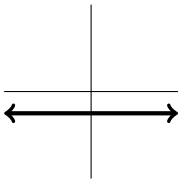
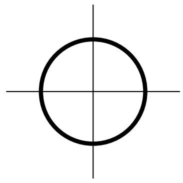
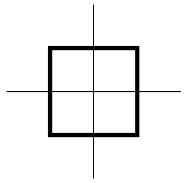
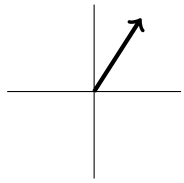
176. Write $(5e^{70^\circ i})(2e^{-40^\circ i})$ in exponential form and polar form and rectangular form.

177. Write $(3+2i)(3-2i)$ in rectangular form.

☆178. Which of the following is equal to i^i?

(A) $\dfrac{i}{\sqrt{2}}$ (B) $\ln(2)+i$ (C) $\dfrac{1}{\sqrt{e^\pi}}$ (D) $e^{\sqrt{3}}$ (E) $2\pi i$ (F) $\dfrac{\ln(\pi)}{2}$

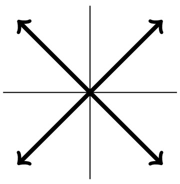
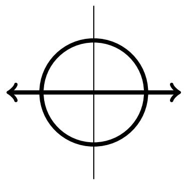
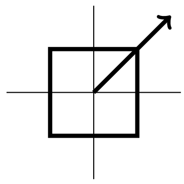
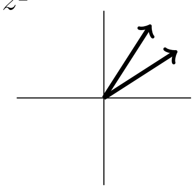
179. Which of the following shows all complex numbers for which $|z|=1$?

(A)  (B)  (C)  (D) 

180. Which of the images from #179 shows all complex numbers with $\arg(z)=1$?

181. Which of the images from #179 shows all complex numbers for which $z+i$ is real (meaning that the imaginary part of $z+i$ is zero)?

☆182. Which of the following shows all complex numbers for which $\dfrac{1}{1+z^2}$ is real?

(A)  (B)  (C)  (D) 

The **conjugate** of the complex number z, written as $\bar{z}$ and spoken as "Z bar", is the reflection of z over the real-axis. In formulas,

$$\overline{a+bi} = a-bi \qquad \text{and} \qquad \overline{re^{\theta i}} = re^{-\theta i}$$

if a, b, r, θ are real numbers.

183. Given that $\overline{z} = 5+2i$ and $\overline{w} = 3-6i$, calculate $\overline{w+z}$.

184. (a) For $z = \dfrac{\sqrt{7}}{2} + \dfrac{\sqrt{11}}{3}i$, calculate $z+\overline{z}$.

(b) For $z = 31 + \dfrac{\sqrt{3+\pi}}{\log(4)-12}i$, calculate $z+\overline{z}$.

(c) For $z = 9e^{(\pi/8)i}$, calculate $z\cdot\overline{z}$.

(d) For $z = \sqrt{26}e^{(8e^3-\sqrt{5})i}$, calculate $z\cdot\overline{z}$.

185. How are $|z|$ and $|\overline{z}|$ related? How are $\arg(z)$ and $\arg(\overline{z})$ related?

186. (a) Give an example of a number z for which $z+\overline{z}=-12$, or explain why no such z can exist.

(b) Give an example of a number z for which $z\cdot\overline{z}=-12$, or explain why no such z can exist.

A **polynomial in x** is a function that can be written in the form

$$\_\_x^n + \_\_x^{n-1} + \cdots + \_\_x^2 + \_\_x + \_\_,$$

where each blank—called a **coefficient**—is a real or complex number, possibly including zero. A **real polynomial** is one whose coefficients are real numbers. In general, variables other than x can also be used (when complex numbers are involved, it is common, but *not* required, to use the variable z).

The **degree** of $f(x)$ is the highest power of x that has a non-zero coefficient.

187. Which of the following are polynomials (in any variable)?
   (a) $8x^2 + 4x + 1$
   (b) $8z^2 + 4z + 1$
   (c) $x^{10} + 5x^6 - 100x$
   (d) $(x^5 - 2x + 1)(x + 1)$
   (e) $(x^5 - 2x + 1)\sin(x)$
   (f) $3x^2 + 3x^{1/2} - 4$
   (g) $x^2 + 2^x$

188. Which of the following are real polynomials (in any variable)?
   (a) $8x^2 + 4x + 1$
   (b) $8z^2 + 4z + 1$
   (c) $z^2 + 1$
   (d) $z^2 + i$
   (e) $(2 + i)x + (4 - i)$
   (f) $(z + i)(z - i)$

189. For each of the following, give the degree if the expression is a polynomial in x, and otherwise write "not a polynomial".
   (a) $\frac{5}{2}x^3 - 7x + 8$
   (b) $9x^{10}$
   (c) $6x^5 + \frac{1}{3}x + 5x^{-2}$
   (d) $3x^2 + \sin(x)$
   (e) $(x^2 + 2x - 1)^3$
   (f) $5x$
   (g) 5
   ☆(h) 0
   (i) $\dfrac{8x + 1}{2x}$
   (j) $\dfrac{x^3 + 7x}{2}$

The number c is a **zero** (also called a **root**) of the polynomial $f(x)$ if $f(c) = 0$.

190. Find all the zeroes of $2x^2 + x - 15$.

191. A cannonball fired at 400 m/s at an angle of 52° will have an initial vertical velocity of $400\sin(52°) \approx 315.2$ m/s, and it will have a height of
$$h(t) = \tfrac{-9.8}{2}t^2 + 315.2\,t$$
meters after t seconds. How many seconds will it take for the cannonball to reach the ground?

192. Find all the roots of $x^5 - 6x^4 + 34x^3$.

193. Given that 4 is one zero of $z^3 - 4z^2 + 49z - 196$, find all its roots.